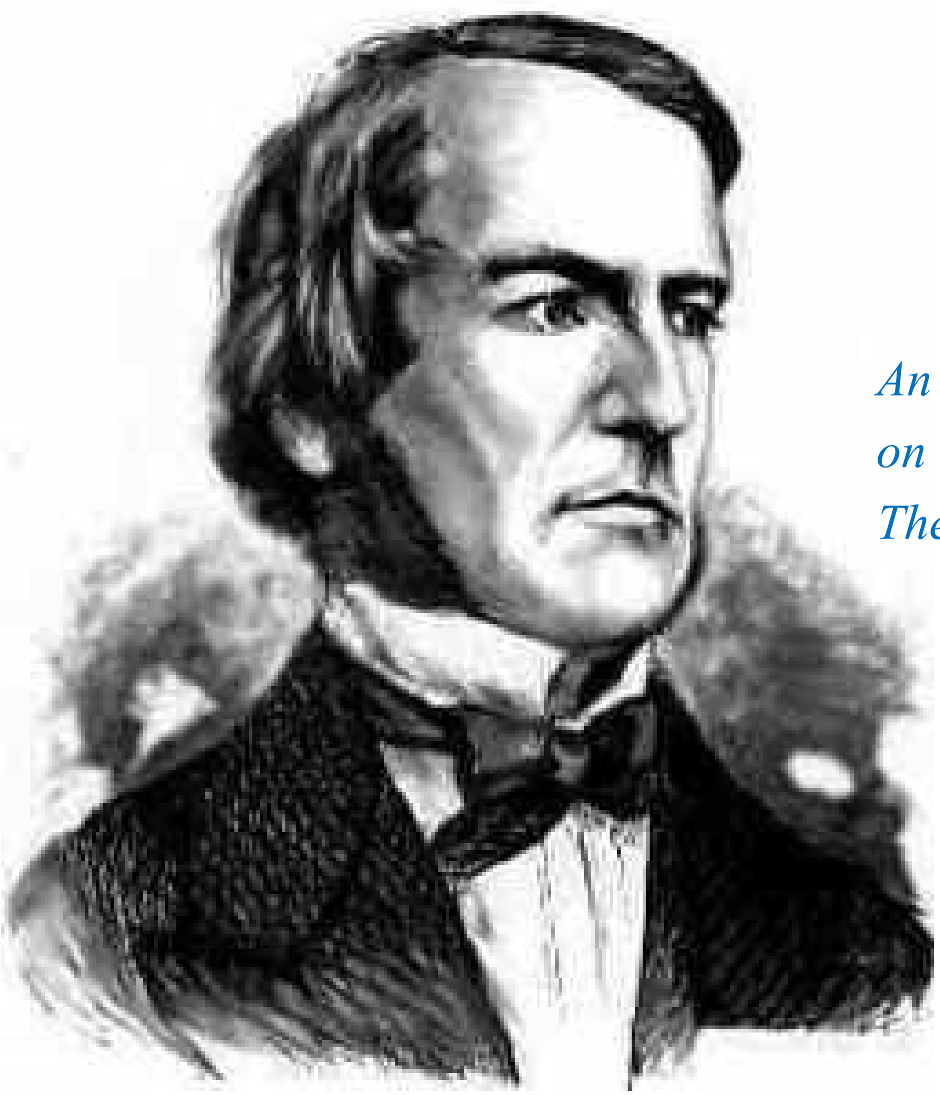
Combinational Logic Circuits

- ✓ Part 1 – Gate Circuits and Boolean Equations
  - Binary Logic and Gates
  - Boolean Algebra
  - Standard Forms
- ✓ Part 2 – Circuit Optimization
- ✓ Part 3 – Additional Gates and Circuits

Binary Logic and Gates

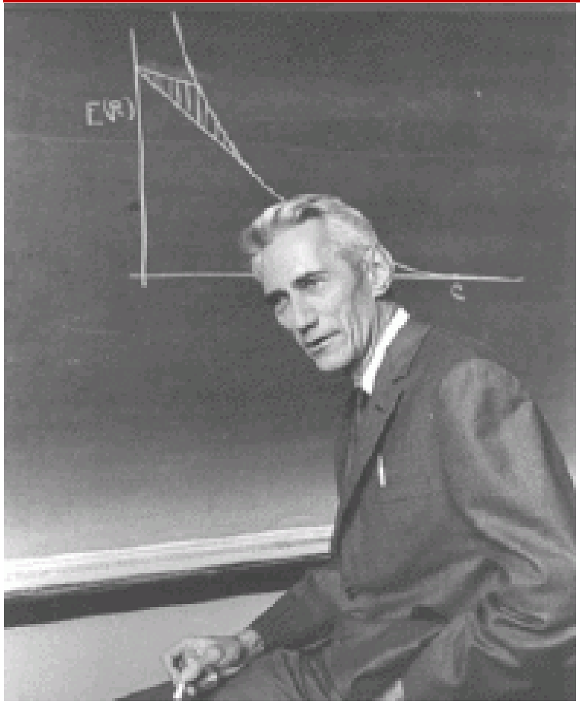
- ✓Binary variables take on one of two values.
- ✓Logical operators operate on binary values and binary variables.
- ✓Basic logical operators are the logic functions AND, OR and NOT.
- ✓Logic gates implement logic functions.
- ✓Boolean Algebra: a mathematical system for specifying and transforming logic functions.
- ✓We study Boolean algebra as a foundation for DESIGNING AND ANALYZING DIGITAL SYSTEMS!

George Boole (1815-1864)

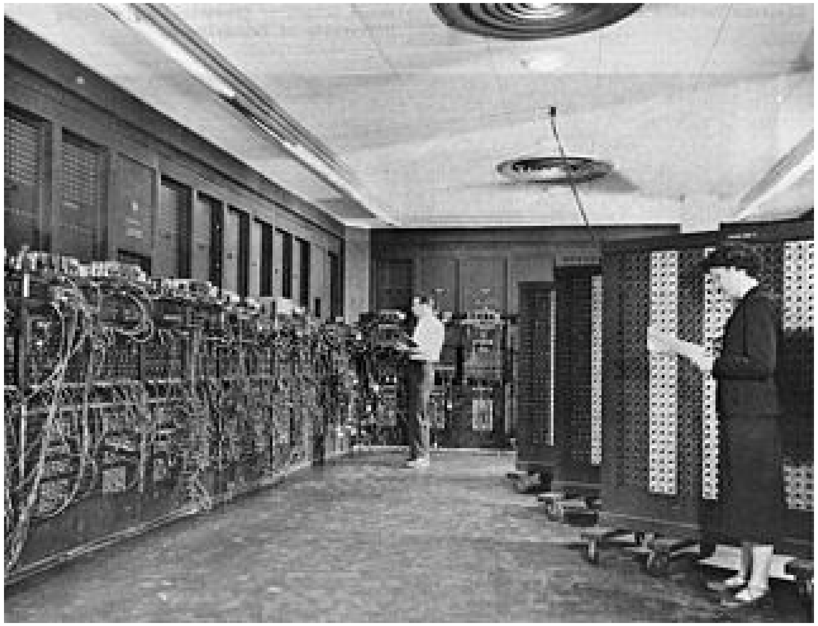


An Investigation of the Laws of Thought, on Which are founded the Mathematical Theories of Logic and Probabilities (1854)

Claude Shannon (1916-2001)



ENIAC (1946)
(Electronic
Numerical
Integrator
And
Calculator)



A Symbolic Analysis of Relay and Switching Circuits (1938)

Binary Variables

- ✓ Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- ✓ We use 1 and 0 to denote the two values.
- ✓ Variable identifier examples:
  - A, B, y, z, or X_1 for now
  - RESET, START_IT, or ADD1 later

Boolean Functions

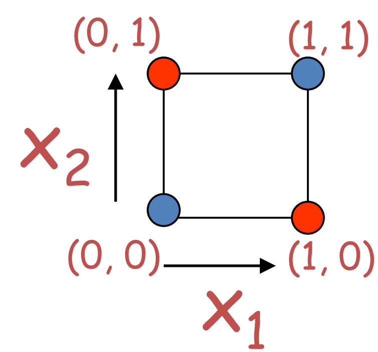
Boolean Function: $\underbrace{f(x_1,\ldots,x_n)}_{\text{arguments}}:\ \underbrace{\{0,1\}^n}_{\text{domain}} \longrightarrow \underbrace{\{0,1\}}_{\text{codomain}}$

- $x_1, x_2, \ldots$ are variables
- $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots$ are literals
- essentially: f maps each vertex of B^n to 0 or 1

Example:

$$f = \{((x_1 = 0, x_2 = 0), 0), ((x_1 = 0, x_2 = 1), 1),$$
$$((x_1 = 1, x_2 = 0), 1), ((x_1 = 1,\ x_2 = 1), 0)\}$$

$$\begin{array}{c|c|c|} & \multicolumn{2}{c}{x_2} \\ \cline{2-3} x_1 & 0 & 1 \\ \cline{2-3} & 1 & 0 \\ \cline{2-3} \end{array}$$



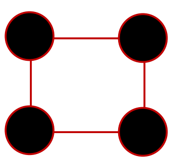
The Boolean n-cube B^n

- ✓ B^1 (B) = {0,1}
- ✓ B^2 = {0,1} x {0,1} = {00, 01, 10, 11}
- ✓ Arrangement of function table on a hypercube
  - The function value f_j is adjacent in each dimension of the hypercube to f_k where K is obtained from j by complementing one and only one input variable: $X_0 X_1 \ldots X_n$ is adjacent to

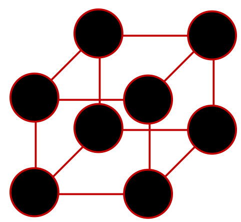
$$\overline{X}_0 X_1 \ldots X_n$$
$$X_0 \overline{X}_1 \ldots X_n$$
$$\ldots$$
$$X_0 X_1 \ldots \overline{X}_n$$

B^1

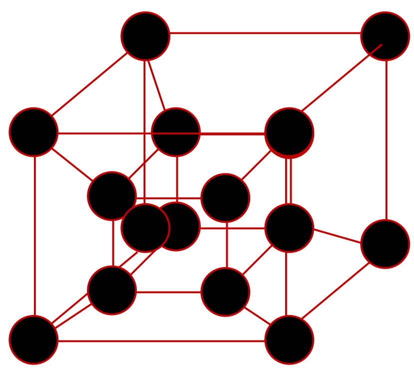
B^2



B^3



B^4

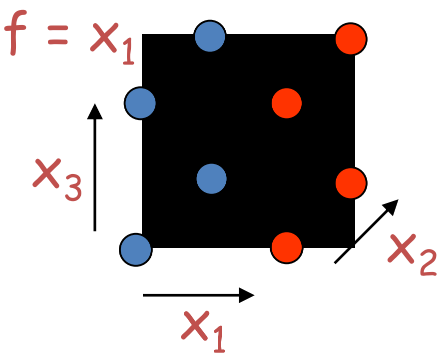


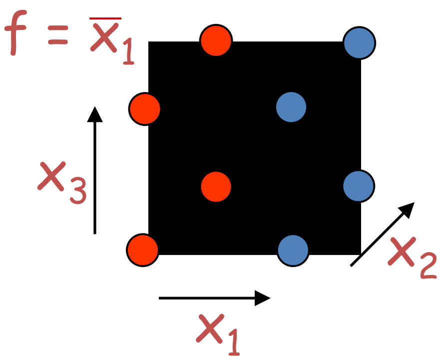
Boolean Functions

- The Onset of f is $\{x \mid f(x) = 1\} = f^{-1}(1) = f^1$
- The Offset of f is $\{x \mid f(x) = 0\} = f^{-1}(0) = f^0$

- if $f^1 = B^n, f$ is the tautology. i.e. $f \equiv 1$
- if $f^0 = B^n \ (f^1 = \varnothing)$, f is not satisfyable, i.e. $f \equiv 0$
- if $f(x) = g(x) \ \textit{for all} \ x \in B^n$, then f and g are equivalent
- we say f instead of f^1

- literals: A literal is a variable or its negation x, $\bar{x}$ and represents a logic function





Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT
- AND is denoted by a dot (•).
- OR is denoted by a plus (+).
- NOT is denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable.
- The order of evaluation in a Boolean expression is:
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- Consequence: Parentheses appear around OR expressions
- Example: $F = A(B + C)(C + D)$

Fundamentals of Boolean Algebra

✓ *Basic Postulates*

- ***Postulate 1 (Definition)***: A Boolean algebra is a closed algebraic system containing a set K of two or more elements and the two operators $\bullet$ and $+$.
- ***Postulate 2 (Existence of 1 and 0 element)***:

  (a) $a + 0 = a$ (identity for $+$), (b) $a \bullet 1 = a$ (identity for $\bullet$)
- ***Postulate 3 (Commutativity)***:

  (a) $a + b = b + a$ (b) $a \bullet b = b \bullet a$
- ***Postulate 4 (Associativity)***:

  (a) $a + (b + c) = (a + b) + c$ (b) $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
- ***Postulate 5 (Distributivity)***:

  (a) $a + (b \bullet c) = (a + b) \bullet (a + c)$ (b) $a \bullet (b + c) = a \bullet b + a \bullet c$
- ***Postulate 6 (Existence of complement)***:
- (a) $\overline{a} + a = 1$ (b) $\overline{a} \bullet a = 0$

Normally $\bullet$ is omitted. A switching algebra is a BA with F={0,1}

Notation Examples

- ✓ Examples:
  - $Y = A \bullet B = A\ B$ is read "Y is equal to A AND B."
  - $z = x + y$ is read "z is equal to x OR y."
  - $X = \overline{A}$ is read "X is equal to NOT A."

- Note: The statement:

  $1 + 1 = 2$ (read "one plus one equals two")

  is not the same as

  $1 + 1 = 1$ (read "1 or 1 equals 1").

Operator Definitions

- ✓ Operations are defined on the values "0" and "1" for each operator:

AND

X	Y	$Z=X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR

X	Y	$Z=X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT

X	$Z=\overline{X}$
0	1
1	0

Properties of Identities

✓ Some properties:

- ***Idempotence*** (a) $a + a = a$ (b) $a \bullet a = a$
- ***Existence of 0 and 1*** (a) $a + 1 = 1$ (b) $a \bullet 0 = 0$
- ***Involution*** (a) $\overline{\overline{a}} = a$
- ***DeMorgan's*** (a) $\overline{a + b} = \overline{a} \bullet \overline{b}$ (b) $\overline{a \bullet b} = \overline{a} + \overline{b}$

Some Properties of Boolean Algebra

- The dual of an algebraic expression is obtained by interchanging + and • and interchanging 0's and 1's.
- Unless it happens to be self-dual, the dual of an expression does not equal the expression itself.
- Example: $F = (A + \overline{C}) \cdot B + 0$

  dual $F = ((A \cdot \overline{C}) + B) \cdot 1 = A \cdot \overline{C} + B$
- Example: $G = X \cdot Y + \overline{(W + Z)}$

  dual $G = (X+Y) \cdot \overline{(W \cdot Z)} = (X+Y) \cdot (\overline{W} + \overline{Z})$
- Example: $H = A \cdot B + A \cdot C + B \cdot C$

  dual $H = (A + B)(A + C)(B + C) = (A + AC + BA + BC)(B + C)$

  $= (A + BC)(B+C) = AB + AC + BC$. So H is self-dual.
- Are any of these functions self-dual?

Generalized De Morgan's theorems

$$\overline{X_1 X_2 \ldots X_n} = \overline{X_1}+\overline{X_2}+\ldots+\overline{X_n}$$

$$\overline{X_1+X_2+\ldots+X_n} = \overline{X_1}\,\overline{X_2} \ldots \overline{X_n}$$

✓ Proof Generalized De Morgan's theorems by general induction:

Two steps:

- Show that the statement is true for two variables
- Show that if is true for n variable , than is also true for n+1 variables:

Let $Z= X_1 + X_2 +\ldots+ X_n$

$\overline{(X_1 + X_2 +\ldots+ X_n + X_{n+1})} = \overline{(Z + X_{n+1})} = (\overline{Z} \bullet \overline{X_{n+1}}) =$

$(\overline{X_1} \bullet \overline{X_2} \bullet \ldots \bullet \overline{X_n}) \bullet \overline{X_{n+1}}$ by induction hypothesis

Others Properties of Boolean Algebra

- ✓ There can be more that 2 elements other than 1 and 0.
- ✓ What are some common useful Boolean algebras with more than 2 elements?

1. Algebra of Sets
2. Algebra of n-bit binary vectors
3. Quantified Boolean Algebra (QBA)

- ✓ If B contains only 1 and 0, then B is called the Switching Algebra which is the algebra we use most often.

Quantified Boolean formulas (QBFs)

- ✓ Generalize (quantifier-free) Boolean formulas with the additional universal and existential quantifiers: $\forall$ and $\exists$, respectively.
- ✓ In writing a QBF, we assume that the precedences of the quantifiers are lower than those of the Boolean connectives.
- ✓ In a QBF, variables being quantified are called bound variables, whereas those not quantified are called free variables.
- ✓ Any QBF can be rewritten as a quantifier-free Boolean formula through quantifier elimination by formula expansion (among other methods), e.g.,

$$\forall x{:}f(x; y) = f(0; y) \bullet f(1; y)$$

and

$$\exists x{:}f(x; y) = f(0; y) + f(1; y)$$

- ✓ Consequently, for any QBF φ, there exists an equivalent quantifier-free Boolean formula that refers only to the free variables of φ.
- ✓ QBFs are thus of the same expressive power as quantifier-free Boolean formulas, but can be more succinct.

Boolean Algebraic Proofs: Example 1

✓ $A + A \cdot B = A$ **Absorption Theorem**

Proof Steps	Justification	
$A + A \cdot B$		
$= A \cdot 1 + A \cdot B$	$X = X \cdot 1$	Identity for •
$= A \cdot (1 + B)$	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$	Distributive Law
$= A \cdot 1$	$1 + X = 1$	Existence of 1
$= A$	$X \cdot 1 = X$	Identity for •

Example 2: Boolean Algebraic Proofs

✓ $AB + \overline{A}C + BC = AB + \overline{A}C$ Consensus Theorem

Proof Steps	Justification
$AB + \overline{A}C + BC$	
$= AB + \overline{A}C + 1 \cdot BC$	Identity for •
$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$	Existence of complement
$= AB + \overline{A}C + ABC + \overline{A}BC$	Distributive Law
$= AB \cdot (1 + C) + \overline{A}C \cdot (1 + B)$	Distributive Law
$= AB + \overline{A}C$	Existence of 1

✓ $(A+B) \cdot (\overline{A}+C) \cdot (B+C) = (A+B) \cdot (\overline{A}+C)$ Dual identity

Useful Theorems

- ✓ $X \cdot Y + \overline{X} \cdot Y = Y$ $\quad (X + Y) \cdot (\overline{X} + Y) = Y$ **Minimization**
- ✓ $X + X \cdot Y = X$ $\quad X \cdot (X + Y) = X$ **Absorption**
- ✓ $X + \overline{X} \cdot Y = X + Y$ $\quad X \cdot (\overline{X} + Y) = X \cdot Y$ **Simplification**
- ✓ $X \cdot Y + \overline{X} \cdot Z + Y \cdot Z = X \cdot Y + \overline{X} \cdot Z$ **Consensus**

  $(X + Y) \cdot (\overline{X} + Z) \cdot (Y + Z) = (X + Y) \cdot (\overline{X} + Z)$
- ✓ $\overline{X + Y} = \overline{X} \cdot \overline{Y}$ $\quad \overline{X \cdot Y} = \overline{X} + \overline{Y}$ **De Morgan's Law**

Example 3: Boolean Algebraic Proofs

✓ $\overline{(X + Y)}Z + X\overline{Y} = \overline{Y}(X + Z)$

Proof Steps	Justification	
$\overline{(X + Y)}Z + X\overline{Y}$		
$= X'\,Y'\,Z + X\,Y'$	$(A + B)' = A' \cdot B'$	De Morgan's Law
$= Y'\,X'\,Z + Y'\,X$	$A \cdot B = B \cdot A$	Commutative Law
$= Y'\,(X'\,Z + X)$	$A(B + C) = AB + AC$	Distributive Law
$= Y'\,(X' + X)(Z + X)$	$A + BC = (A + B)(A + C)$	Distributive Law
$= Y' \cdot 1 \cdot (Z + X)$	$A + A' = 1$	Existence of complement
$= Y'\,(X + Z)$	$1 \cdot A = A,\ A + B = B + A$	Commutative Law

Expression Simplification

- ✓ An application of Boolean algebra
- ✓ Simplify to contain the smallest number of literals (complemented and un-complemented variables):

$$AB+\bar{A}CD+\bar{A}BD+\bar{A}C\bar{D}+ABCD$$

15 literal, 2 levels

$= AB+ABCD+\bar{A}CD+\bar{A}C\bar{D}+\bar{A}BD$

$= AB+AB(CD)+\bar{A}C(D+\bar{D})+\bar{A}BD$

$= AB+\bar{A}C+\bar{A}BD = B(A+\bar{A}D)+\bar{A}C$

$= B(A+D)+\bar{A}C \quad = \quad BA + BD + \bar{A}C$

5 literal, 3 levels — 6 literals, 2 levels

Complementing Functions

- ✓ Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- ✓ Example: Complement $F = \overline{x}y\overline{z} + x\overline{y}\overline{z}$

  $\overline{F} = (x + \overline{y} + z)(\overline{x} + y + z)$
- ✓ Example: Complement $G = (\overline{a} + bc)\overline{d} + e$

$$\overline{G} = (a(\overline{b} + \overline{c})) + d)\overline{e} = (a(\overline{b} + \overline{c}) + d)\overline{e}$$

Boolean Function Evaluation

$F1 = xy\bar{z}$

$F2 = x + \bar{y}z$

$F3 = \bar{x}\bar{y}\bar{z} + \bar{x}\,y\,z + x\bar{y}$

$F4 = x\bar{y} + \bar{x}\,z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0		
0	0	1	0	1		
0	1	0	0	0		
0	1	1	0	0		
1	0	0	0	1		
1	0	1	0	1		
1	1	0	1	1		
1	1	1	0	1		

Boolean Function Evaluation

$F1 = xy\bar{z}$
$F2 = x + \bar{y}z$
$F3 = \bar{x}\bar{y}\bar{z} + \bar{x}yz + x\bar{y}$
$F4 = x\bar{y} + \bar{x}z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0	1	0
0	0	1	0	1	0	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

Shannon Expansion

- Let $f: B^n \rightarrow B$ be a Boolean function, and $x=(x_1,x_2, \dots, x_n)$ the variables in the support of f. The cofactor (residual) f_a of f by a literal $a=x_i$ or $a=\overline{x}_i$ is:

$$f_{x_i}(x_1, x_2, \dots, x_n) = f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

$$f_{\overline{x}_i}(x_1, x_2, \dots, x_n) = f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$

- Shannon theorem:

$$f = x_i f_{x_i} + \overline{x}_i f_{\overline{x}_i}$$

$$f = [x_i + f_{\overline{x}_i}]\,[\overline{x}_i + f_{x_i}]$$

- We say that f is expanded about x_i. x_i is called the splitting variable.

Boolean difference

- ✓ Universal and existential quantifications can be expressed in terms of cofactor, with

$$\forall x_i.f = f_{x_i} \cdot f_{\overline{x_i}} \text{ and } \exists x_i.f = f_{x_i} + f_{\overline{x_i}}$$

- ✓ Moreover, the Boolean difference $\partial f/\partial x_i$ of f with respect to variable x_i is defined as

$$\partial f/\partial x_i = \overline{f_{x_i} \equiv f_{\overline{x_i}}} = f_{x_i} \oplus f_{\overline{x_i}}$$

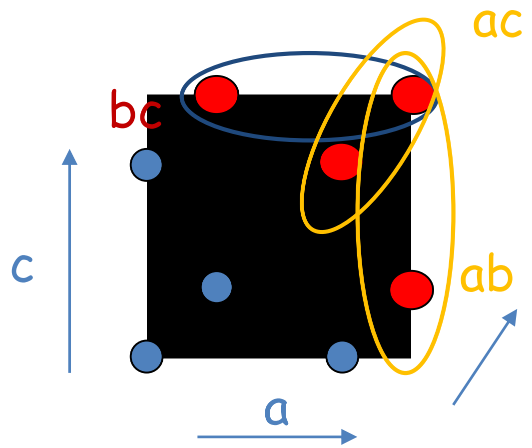
  where $\oplus$ denotes an *exclusive-or (xor) operator.*
- ✓ Using the Boolean difference operation, we can tell whether a Boolean function functionally depends on a variable. If $\partial f/\partial x_i$ equals constant 0, then the valuation of f does not depend on x_i, that is, x_i is a redundant variable for f.
- ✓ We call that x_i is a *functional support variable* of f if x_i is not a redundant variable.

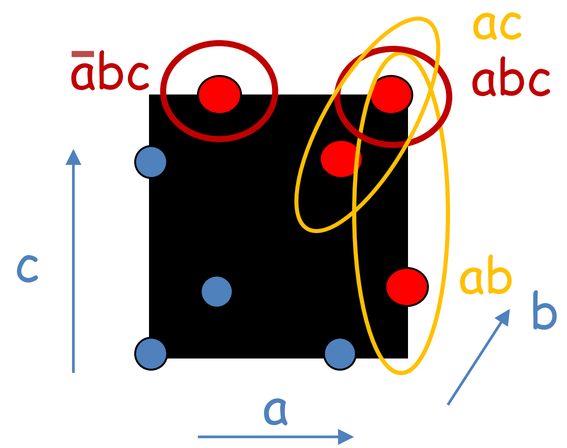
Example

$F = ab + ac + bc$

$F = a\, F_a + \bar{a}\, F_{\bar{a}}$

$F = ab + ac + abc + \bar{a}bc$





Cube bc got split into two cubes $\bar{a}bc$ and abc

Representation of Boolean Functions

- ✓ We need representations for Boolean Functions for two reasons:
  - to represent and manipulate the actual circuit we are "synthesizing"
  - as mechanism to do efficient Boolean reasoning
- ✓ Forms to represent Boolean Functions
  - Truth table
  - List of cubes (Sum of Products, Disjunctive Normal Form (DNF))
  - List of conjuncts (Product of Sums, Conjunctive Normal Form (CNF))
  - Boolean formula
  - Binary Decision Tree, Binary Decision Diagram
  - Circuit (network of Boolean primitives)

Truth Tables

- *Truth table* – a tabular listing of the values of a function for all possible combinations of values on its arguments
- Example: Truth tables for the basic logic operations:

AND		
X	Y	$Z=X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR		
X	Y	$Z=X+Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT	
X	$Z=\overline{X}$
0	1
1	0

Truth Table

- Truth table (Function Table):

  The truth table of a function $f : B^n \rightarrow B$ is a tabulation of its value at each of the 2^n vertices of B^n.

- In other words the truth table lists all minterms

  Example: $f = abc\bar{d} + ab\bar{c}\bar{d} + a\bar{b}c\bar{d} + \bar{a}bc\bar{d} + \bar{a}b\bar{c}\bar{d} + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}\bar{c}\bar{d}$

  The truth table representation is

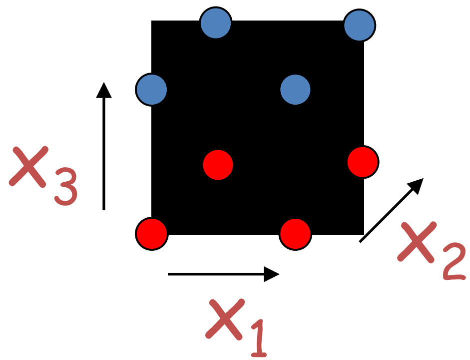
  - intractable for large n
  - canonical

  Canonical means that if two functions are the same, then the canonical representations of each are isomorphic.

	abcd	f
0	0000	1
1	0001	1
2	0010	1
3	0011	0
4	0100	1
5	0101	0
6	0110	1
7	0111	0

	abcd	f
8	1000	0
9	1001	0
10	1010	1
11	1011	0
12	1100	1
13	1101	0
14	1110	1
15	1111	0

✓Truth Table or Function table



$x_1x_2x_3$	
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	0
1 0 0 $\Rightarrow$	1
1 0 1	0
1 1 0	1
1 1 1	0

- ✓ There are 2^n vertices in input space B^n
- ✓ There are 2^{2^n} distinct logic functions.
  - Each subset of vertices is a distinct logic function: $f \subseteq B^n$

Boolean Formula

✓ A <u>Boolean formula</u> is defined as an expression with the following syntax:

```
formula ::=     '(' formula ')'
          |     <variable>
          |     formula "+" formula       (OR operator)
          |     formula "·" formula       (AND operator)
          |     ~ formula                 (complement)
```

Example:

$$f = (x_1 \cdot x_2) + (x_3) + (x_4 \cdot (\sim x_1))$$

typically the "·" is omitted and the '(' and '~'are simply reduced by priority,

e.g.

$$f = x_1x_2 + x_3 + x_4 \sim x_1 = x_1x_2 + x_3 + x_4\overline{x}_1$$

Cubes

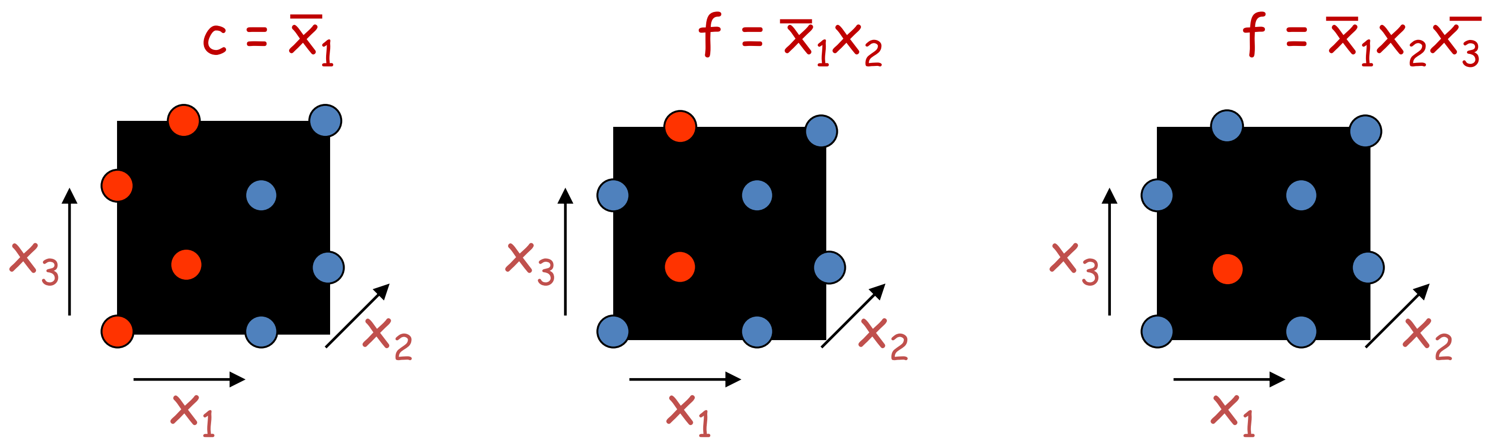
- ✓ A cube is defined as the AND of a set of literal functions ("conjunction" of literals).

Example:

$$C = \bar{x}_1 x_2 \bar{x}_3$$

represents the following function

$$f = (x_1{=}0)(x_2{=}1)(x_3{=}0)$$



Cubes

- ✓ If $C \subseteq f$, C a cube, then C is an implicant of f.
- ✓ If $C \subseteq B^n$, and C has k literals, then $|C|$ covers 2^{n-k} vertices.

  Example:

  $C = x\bar{y} \subseteq B^3$

  $k = 2$, $n = 3$ => $|C| = 2 = 2^{3-2}$.

  $C = \{100, 101\}$

- ✓ An implicant with n literals is a minterm.

List of Cubes

✓ Sum of Products (SOP):

- A function can be represented by a sum of products (cubes):

  $f = ab + ac + bc$

  Since each cube is a product of literals, this is a "sum of products" (SOP) representation

- A SOP can be thought as a set of cubes F

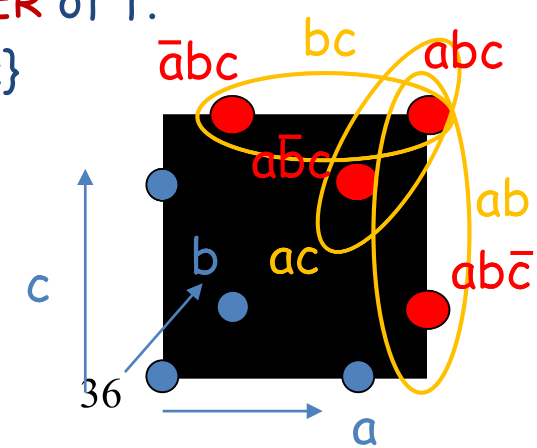
  $F = \{ab, ac, bc\}$

- A set of cubes that represents f is called a COVER of f.

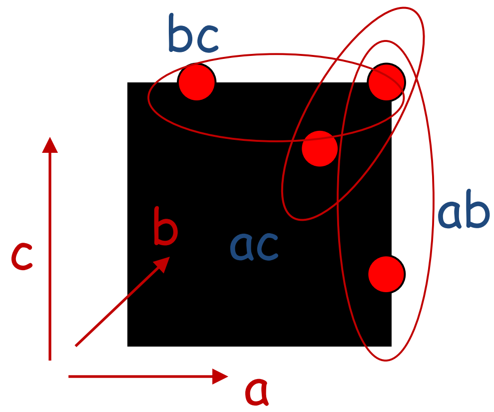
  $F_1=\{ab, ac, bc\}$ and $F_2=\{abc, ab\bar{c}, a\bar{b}c, \bar{a}bc\}$

  are covers of

  $f = ab + ac + bc.$



SOP



= onset minterm

Note that each onset minterm is "covered" by at least one of the cubes, and covers no offset minterm.

- ✓ Covers (SOP's) can efficiently represent many logic functions (i.e. for many, there exist small covers).
- ✓ Two-level minimization seeks the minimum size cover (least number of cubes)

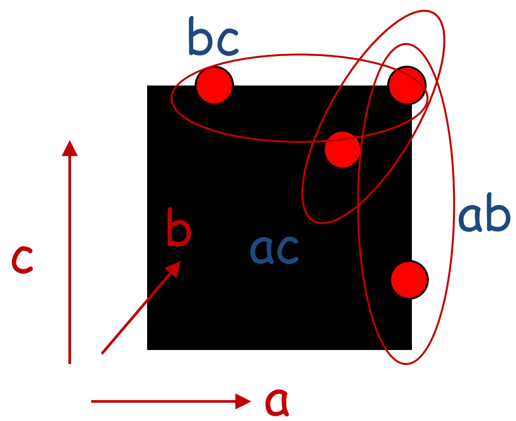
Irredundant

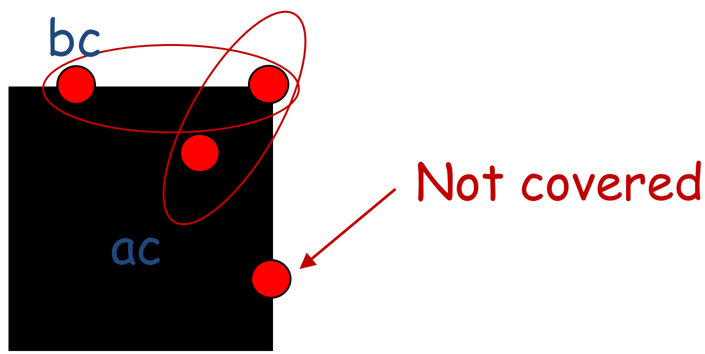
✓ Let $F = \{c_1, c_2, \ldots, c_k\}$ be a cover for f.

$$f = \sum_{i=1}^{k} c_i$$

A cube $c_i \in F$ is IRREDUNDANT if $F \setminus \{c_i\} \neq f$

Example 2: $f = ab + ac + bc$





$F \setminus \{ab\} \neq f$

Prime

- ✓ A literal j of cube $c_i \in F$ ($=f$) is PRIME if

$$(F \setminus \{c_i\}) \cup \{c'_i\} \neq f$$

where c'_i is c_i with literal j of c_i deleted.

- ✓ A cube of F is prime if all its literals are prime.

Example 3

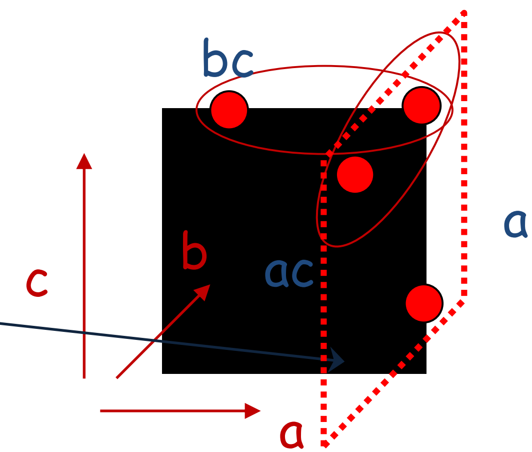
$f = ab + ac + bc$

$c_i = ab$; $c'_i = a$ (literal b deleted)

$F \setminus \{c_i\} \cup \{c'_i\} = a + ac + bc$

$F = ac + bc + a =$

$F \setminus \{c_i\} \cup \{c'_i\}$



Not equal to f since this offset vertex is covered

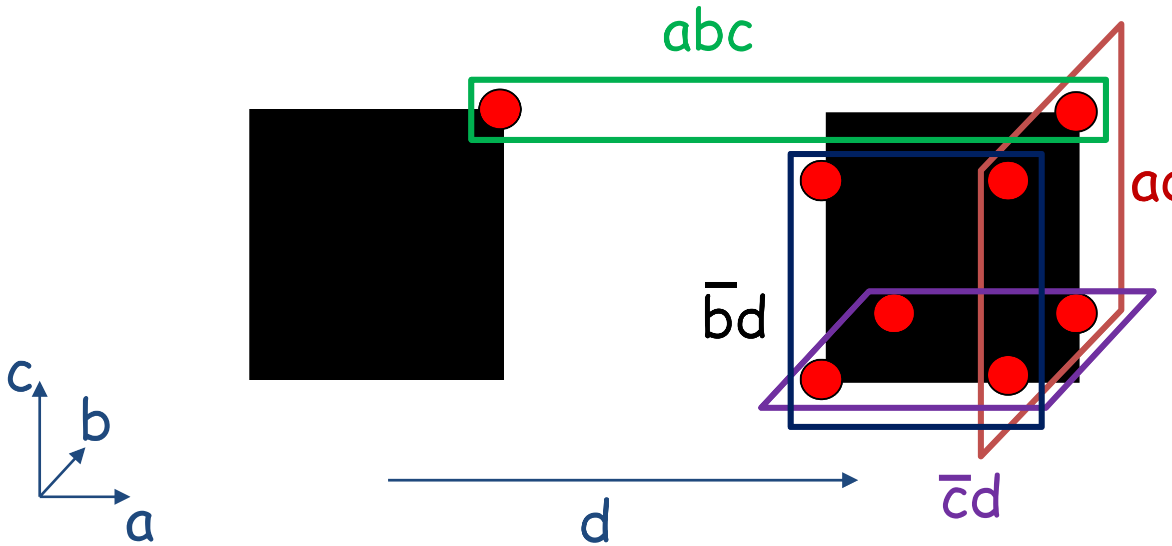
Prime and Irredundant Covers

- ✓ Definition 1 A cover is prime (irredundant) if all its cubes are prime (irredundant).
- ✓ Definition 2 A prime of f is essential (essential prime) if there is a minterm (essential vertex) in that prime but in no other prime.

Prime and Irredundant Covers

$f = abc + \bar{b}d + \bar{c}d$ is prime and irredundant.

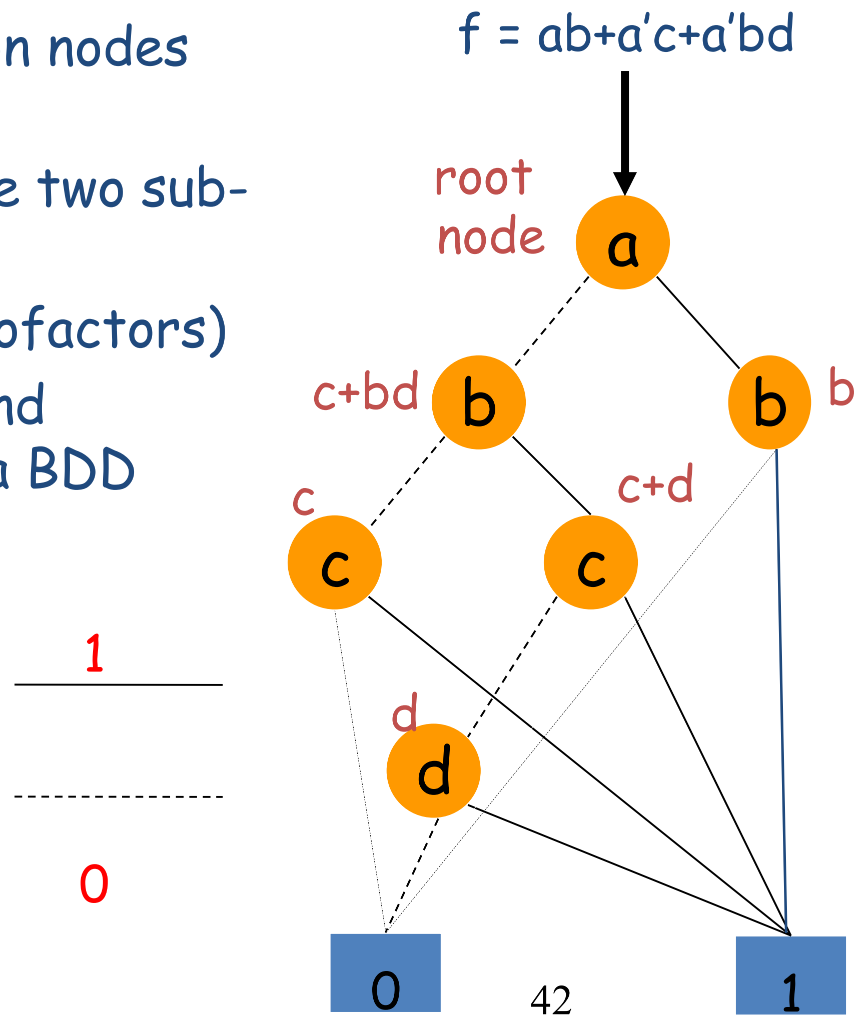
abc is essential since $abc\bar{d} \in abc$, but not in $\bar{b}d$ or $\bar{c}d$ or ad



What other cube is essential? What prime is not essential?

Binary Decision Diagram (BDD)

- ✓ Graph representation of a Boolean function f
  - vertices represent decision nodes for variables
  - two children represent the two sub-functions
    - f(x = 0) and f(x = 1) (cofactors)
  - restrictions on ordering and reduction rules can make a BDD representation canonical



Logic Functions

- ✓ There are infinite number of equivalent logic formulas

$$
\begin{aligned}
f &= x + y \\
&= \overline{x}y + xy + x\overline{y} \\
&= \overline{x}x + x\overline{y} + y \\
&= (x + y)(x + y) + x\overline{y}
\end{aligned}
$$

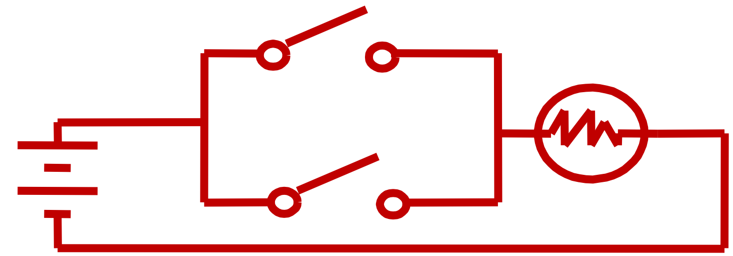
- ✓ Synthesis = Find the best formula (or "representation")

Logic Function Implementation

✓ Using Switches

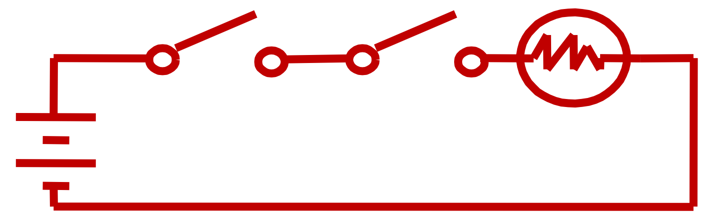
- For inputs:
  - logic 1 is switch closed
  - logic 0 is switch open

Switches in parallel ⇨ OR



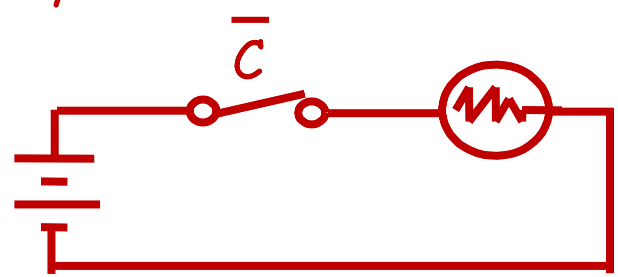
- For outputs:
  - logic 1 is light on
  - logic 0 is light off.

Switches in series ⇨ AND



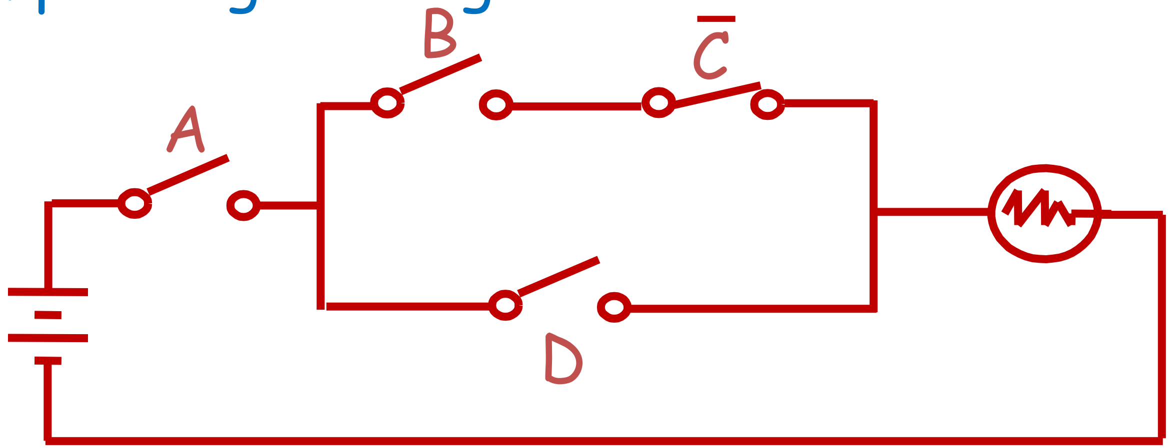
- NOT uses a switch such that:
  - logic 1 is switch open
  - logic 0 is switch closed

Normally-closed switch ⇨ NOT



Logic Function Implementation (Continued)

- ✓ Example: Logic Using Switches



- ✓ Light is on (L = 1) for

  $L(A, B, C, D) = AD + AB\overline{C}$

  and off (L = 0), otherwise.

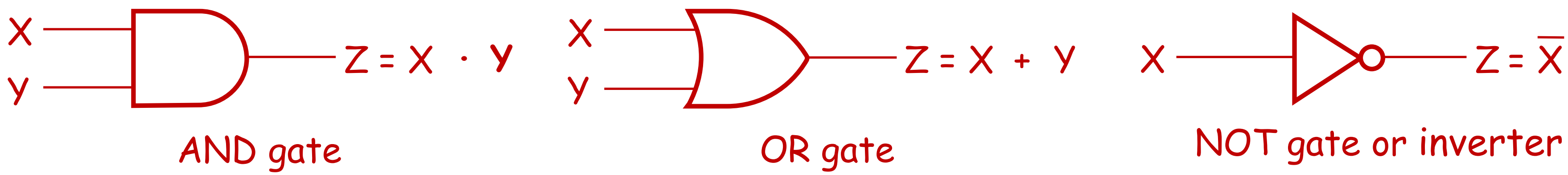
- ✓ Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology

Logic Gates

- ✓ In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in *relays*. The switches in turn opened and closed the current paths.
- ✓ Later, *vacuum tubes* that open and close current paths electronically replaced relays.
- ✓ Today, *transistors* are used as electronic switches that open and close current paths.

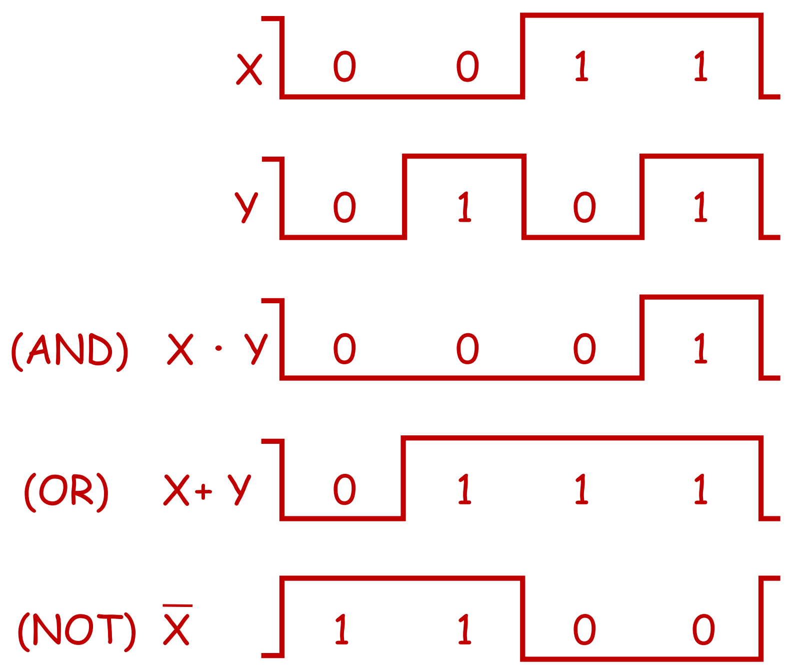
Logic Gate Symbols and Behavior

- ✓ Logic gates have special symbols:



(a) Graphic symbols

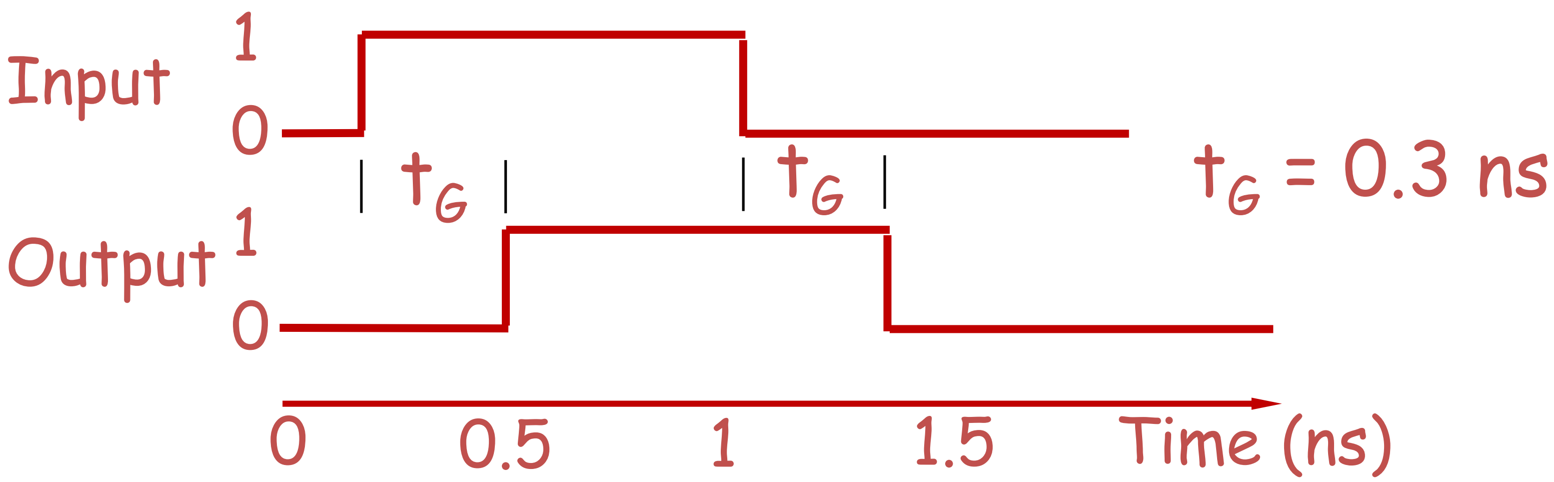
- ✓ And waveform behavior in time as follows:



(b) Timing diagram

Gate Delay

- ✓ In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.
- ✓ The delay between an input change(s) and the resulting output change is the *gate delay* denoted by t_G:



Logic Diagrams and Expressions

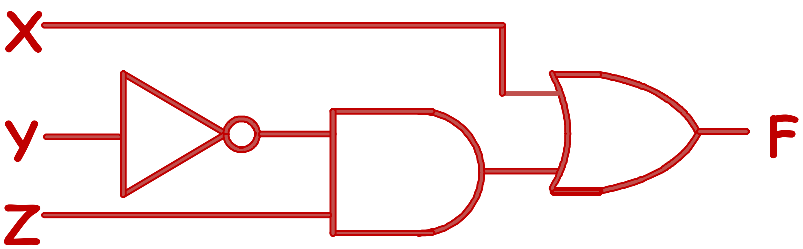
Truth Table

X Y Z	$F = X + \overline{Y} Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Equation

$F = X + \overline{Y} Z$

Logic Diagram



- ✓ Boolean equations, truth tables and logic diagrams describe the same function!
- ✓ Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.

Definitions

Definition:

- A **Boolean circuit** is a directed graph $C(G,N)$ where G are the gates and $N \subseteq G \times G$ is the set of directed edges (nets) connecting the gates.

- Some of the vertices are designated:

  Inputs: $I \subseteq G$

  Outputs: $O \subseteq G$, $I \cap O = \varnothing$

- Each gate g is assigned a Boolean function f_g which computes the output of the gate in terms of its inputs.

Definitions

- The fanout FO(g) of a gate g are all successor vertices of g:

$$FO(g) = \{g' \mid (g,g') \in N\}$$

- The fanin FI(g) of a gate g are all predecessor vertices of g:

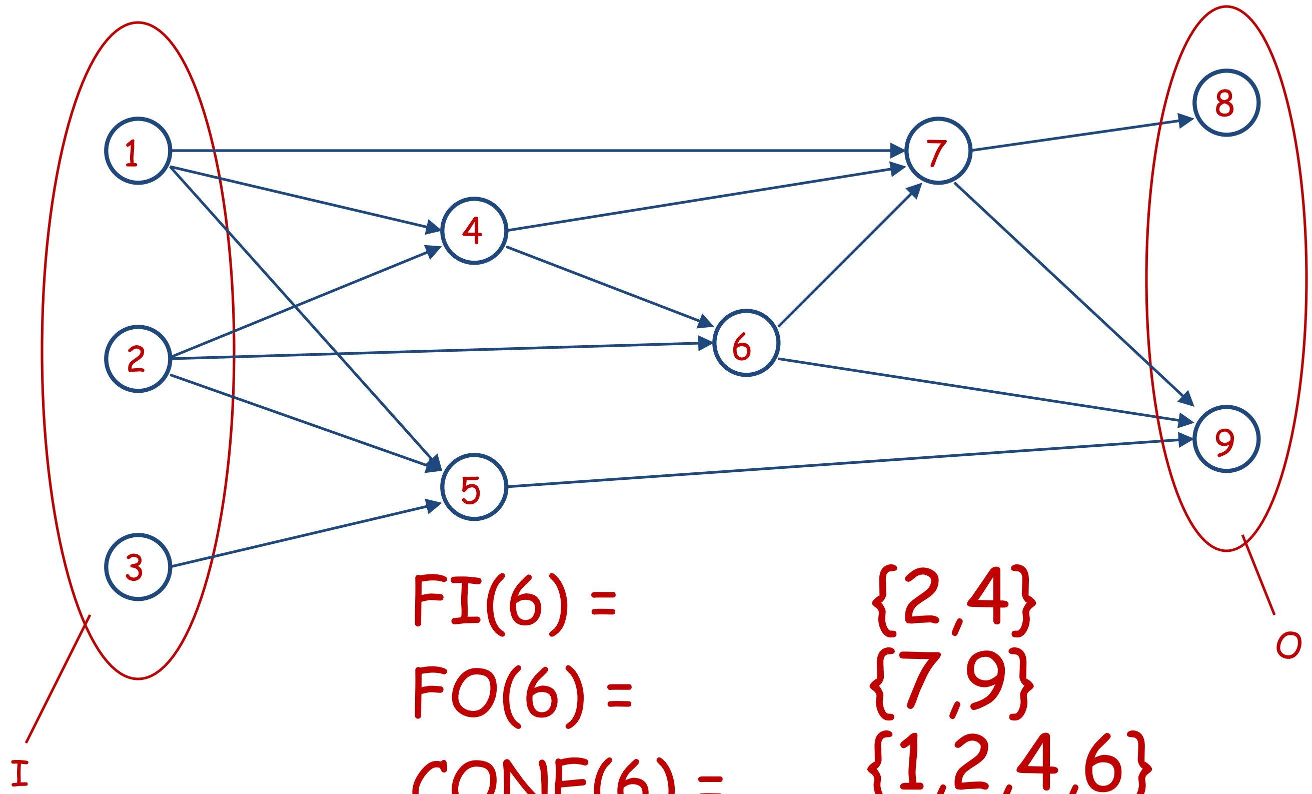
$$FI(g) = \{g' \mid (g',g) \in N\}$$

- The cone CONE(g) of a gate g is the transitive fanin of g and g itself.

- The support SUPPORT(g) of a gate g are all inputs in its cone:

$$SUPPORT(g) = CONE(g) \cap I$$

Example



FI(6) = {2,4}

FO(6) = {7,9}

CONE(6) = {1,2,4,6}

SUPPORT(6) = {1,2}

Circuit Representations

✓ For efficient Boolean reasoning :

- Vertices have fixed number of inputs
- Vertex function is stored as label, well defined set of possible function labels (e.g. OR, AND,OR)
- Circuits are often non-canonical

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has an immediate correspondence to the truth tables
- Canonical Forms in common usage:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)

minterms

- minterms are **AND** terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for n variables.
- Example: Two variables (X and Y)produce $2 \times 2 = 4$ combinations:

  XY (both normal)

  $X\overline{Y}$ (X normal, Y complemented)

  $\overline{X}Y$ (X complemented, Y normal)

  $\overline{X}\,\overline{Y}$ (both complemented)
- Thus there are four minterms of two variables.

Maxterms

- Maxterms are **OR** terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for n variables.
- <u>Example:</u> Two variables (X and Y) produce $2 \times 2 = 4$ combinations:

$X+Y$	(both normal)
$X+\overline{Y}$	(x normal, y complemented)
$\overline{X}+Y$	(x complemented, y normal)
$\overline{X}+\overline{Y}$	(both complemented)

Maxterms and Minterms

- ✓ Examples: Two variable minterms and maxterms.

Index	minterm	Maxterm
0 [00]	$\bar{x}\,\bar{y}$	$x + y$
1 [01]	$\bar{x}\,y$	$x + \bar{y}$
2 [10]	$x\,\bar{y}$	$\bar{x} + y$
3 [11]	$x\,y$	$\bar{x} + \bar{y}$

- ✓ The index is important for describing which variables in the terms are true and which are complemented.

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript (index) is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the **same order** (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms: $(a + b + \bar{c})$, $(a + b + c)$
    - Terms: $(b + a + c)$, $\bar{a}\ c\ b$, and $(c + b + a)$ are NOT in standard order.
  - minterms: $a\ b\ \bar{c}$, $a\ b\ c$, $\bar{a}\ \bar{b}\ c$
    - Terms: $(a + c)$, $b\ c$, and $(\bar{a} + b)$ do not contain **all** variables

Purpose of the Index

- ✓ The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- ✓ For minterms:
  - 1 means the variable is Not Complemented
  - 0 means the variable is Complemented.
- ✓ For Maxterms:
  - 0 means the variable is Not Complemented
  - 1 means the variable is Complemented

Index Example in Three Variables

- ✓ Assume the variables are called X, Y, and Z.
- ✓ The standard order is X, then Y, then Z.
- ✓ The Index 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for minterm 0 ($\overline{x},\overline{y},\overline{z}$) and no variables are complemented for Maxterm 0 (X,Y,Z).
  - minterm 0, called m_0 is $\overline{x}\,\overline{y}\,\overline{z}$.
  - Maxterm 0, called M_0 is $(x + y + z)$.
  - minterm 6 ? $m_6 = x\,y\,\overline{z}$
  - Maxterm 6 ? $M_6 = (\overline{x} + \overline{y} + z)$

Index Examples - Four Variables

Index i	Binary Pattern	minterm m_i	Maxterm M_i
0	0000	$\overline{a}\,\overline{b}\,\overline{c}\,\overline{d}$	$a + b + c + d$
1	0001	$\overline{a}\,\overline{b}\,\overline{c}d$	$a + b + c + \overline{d}$
3	0011	?	?
5	0101	$\overline{a}b\overline{c}d$	$a + \overline{b} + c + \overline{d}$
7	0111	?	$a + \overline{b} + \overline{c} + \overline{d}$
10	1010	$a\overline{b}c\overline{d}$	$\overline{a} + b + \overline{c} + d$
13	1101	?	$\overline{a} + \overline{b} + c + \overline{d}$
15	1111	$abcd$	$\overline{a} + \overline{b} + \overline{c} + \overline{d}$

Minterm and Maxterm Relationship

- ✓ Review: DeMorgan's Theorem
  $\overline{x \cdot y} = \overline{x} + \overline{y}$ and $\overline{x + y} = \overline{x} \cdot \overline{y}$
- ✓ Two-variable example:
  $M_2 = \overline{x} + y$ and $m_2 = x \cdot \overline{y}$
  Thus M_2 is the complement of m_2 and vice-versa.
- ✓ Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables giving:

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Thus M_i is the complement of m_i.

Function Tables for Both

- minterms of 2 variables

x y	m_0 ($\overline{x}\,\overline{y}$)	m_1 ($\overline{x}y$)	m_2 ($x\overline{y}$)	m_3 (xy)
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M_0 ($x+y$)	M_1 ($x+\overline{y}$)	M_2 ($\overline{x}+y$)	M_3 ($\overline{x}+\overline{y}$)
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.

Observations

- ✓ In the function (truth) tables:
  - Each minterm has one and only one 1 present in the $2n$ terms (a minimum of 1s). All other entries are 0.
  - Each Maxterm has one and only one 0 present in the $2n$ terms. All other entries are 1 (a maximum of 1s)
- ✓ We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function
- ✓ We can implement any function by "ANDing" the Maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function
- ✓ This gives us two canonical forms:

  Sum of minterms (SOM) Product of Maxterms (POM)

minterm Function Example

- ✓ Example: Find $F_1 = m_1 + m_4 + m_7$
- ✓ $F1 = \overline{x}\ \overline{y}\ z + x\ \overline{y}\ \overline{z} + x\ y\ z$

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

$$= A'B'C'DE' + A'BC'D'E + AB'C'D'E + AB'CDE$$

Maxterm Function Example

- Example: Implement F1 in maxterms:

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z)\cdot(x + \bar{y} + z)\cdot(x + \bar{y} + \bar{z})$$
$$\cdot(\bar{x} + y + \bar{z})\cdot(\bar{x} + \bar{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F1$
0 0 0	0	$0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0$
0 0 1	1	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
0 1 0	2	$1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$
0 1 1	3	$1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 = 0$
1 0 0	4	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
1 0 1	5	$1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 = 0$
1 1 0	6	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 0 = 0$
1 1 1	7	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

Maxterm Function Example

- ✓ $F(A,B,C,D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- ✓ $F(A, B,C,D) =$

$(A+B+C'+D')(A'+B+C+D)(A'+B+C'+D')(A'+B'+C'+D)$

Canonical Sum of minterms

- Any Boolean function can be expressed as a Sum of minterms.
  - For the function table, the minterms used are the terms corresponding to the 1's
  - For expressions, <u>expand</u> all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term $(v + \overline{v})$
- Example: Implement $f = x + \overline{x}\,\overline{y}$ as a sum of minterms.

  First expand terms: $f = x\,(y + \overline{y}) + \overline{x}\,\overline{y}$

  Then distribute terms: $f = xy + x\overline{y} + \overline{x}\,\overline{y}$

  Express as sum of minterms: $f = m_3 + m_2 + m_0$

Another SOM Example

- ✓ Example: $F = A + \overline{B}\,C$
- ✓ There are three variables, A, B, and C which we take to be the standard order.
- ✓ Expanding the terms with missing variables:

$$F = A(B + B')(C + C') + (A + A')\,B'\,C$$

$$= ABC + ABC' + AB'C + AB'C' + AB'C + A'B'C$$

$$= ABC + ABC' + AB'C + AB'C' + A'B'C$$

$$= m_7 + m_6 + m_5 + m_4 + m_1$$

Standard form $= m_1 + m_4 + m_5 + m_6 + m_7$

Shorthand SOM Form

- ✓ From the previous example, we started with:

$$F = A + \overline{B}\,C$$

- ✓ We ended up with:

$$F = m_1+m_4+m_5+m_6+m_7$$

- ✓ This can be denoted in the formal shorthand:

$$F(A,B,C) = \Sigma m(1,4,5,6,7)$$

- ✓ Note that we explicitly show the standard variables in order and drop the "m" designators.

Canonical Product of Maxterms

- ✓ Any Boolean Function can be expressed as a **Product of Maxterms (POM).**
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law, "ORing" terms missing variable v with a term equal to $v \cdot \overline{v}$ and then applying the distributive law again.
- ✓ Example: Convert to product of maxterms:

$$f(x, y, z) = x + \overline{x}\,\overline{y}$$

Apply the distributive law:

$$x + \overline{x}\,\overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y}$$

Add missing variable z:

$$x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- ✓ Convert to Product of Maxterms:

$$f(A, B, C) = A\,\overline{C} + B\,C + \overline{A}\,\overline{B}$$

- ✓ Use $x + y\,z = (x+y)\cdot(x+z)$ with $x = (A\,\overline{C} + B\,C)$, $y = \overline{A}$, and $z = \overline{B}$ to get:

$$f = (A\,\overline{C} + B\,C + \overline{A})(A\,\overline{C} + B\,C + \overline{B})$$

- ✓ Then use $x + \overline{x}y = x + y$ to get:

$$f = (\overline{C} + BC + \overline{A})(A\,\overline{C} + C + \overline{B})$$

and a second time to get:

$$f = (\overline{C} + B + \overline{A})(A + C + \overline{B})$$

- ✓ Rearrange to standard order,

$f = (\overline{A} + B + \overline{C})(A + \overline{B} + C)$ to give $f = M_2 \cdot M_5$

Function Complements

- **The complement** of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, **the complement** of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x,y,z) = \sum_m(1,3,5,7)$

$$\overline{F}(x,y,z) = \sum_m(0,2,4,6)$$
$$\overline{F}(x,y,z) = \prod_M(1,3,5,7)$$

Conversion Between Forms

- ✓ To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- ✓ Example: Given F as before: $F(x, y, z) = \sum_m (1, 3, 5, 7)$
- ✓ Form the Complement: $\overline{F}(x, y, z) = \sum_m (0, 2, 4, 6)$
- ✓ Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \prod_M (0, 2, 4, 6)$

Standard Forms

- Standard Sum-of-Products (SOP) form: equations are written as an OR of AND terms
- Standard Product-of-Sums (POS) form: equations are written as an AND of OR terms
- Examples:
  - SOP: $A\,B\,C + \overline{A}\,\overline{B}\,C + B$
  - POS: $(A + B)\,(A + \overline{B} + \overline{C})\,C$
- These "mixed" forms are neither SOP nor POS
  - $(A\,B + C)\,(A + C)$
  - $A\,B\,C + A\,C\,(A + B)$

Standard Sum-of-Products (SOP)

- ✓ A sum of minterms form for *n* variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
    - The first level consists of *n*-input AND gates, and
    - The second level is a single OR gate (with fewer than 2^n inputs).
- ✓ This form often can be simplified so that the corresponding circuit is simpler.

Standard Sum-of-Products (SOP)

- A Simplification Example:

$$F(A,B,C) = \Sigma m(1,4,5,6,7)$$

- Writing the minterm expression:

  $F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$

- Simplifying:

  $F = A'\,B'\,C + A\,(B'\,C' + B\,C' + B'\,C + B\,C)$

  $= A'\,B'\,C + A\,(B' + B)\,(C' + C)$

  $= A'\,B'\,C + A \cdot 1 \cdot 1$

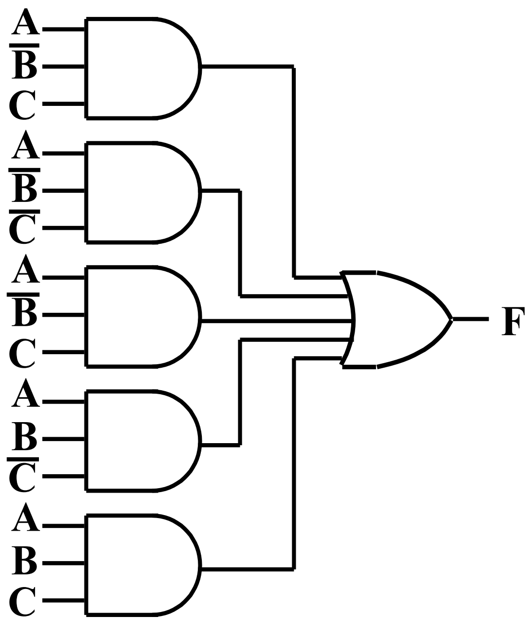
  $= A'\,B'\,C + A$

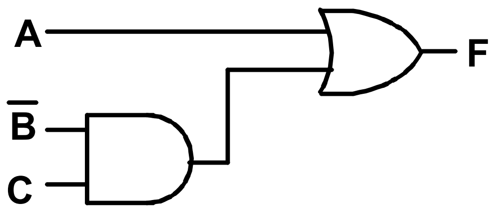
  $= B'C + A$

- Simplified F contains 3 literals compared to 15 in minterm F

AND/OR Two-level Implementation of SOP Expression

- ✓ The two implementations for F are shown below – it is quite apparent which is simpler!





SOP and POS observations

- ✓ The previous examples show that:
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms.
  - Simpler equations lead to simpler two-level implementations
- ✓ Questions:
  - How can we attain a “simplest” expression?
  - Is there only one minimum cost circuit?